

Description of Derivations on Measurable Operator Algebras of Type I

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Abstract

Given a type I von Neumann algebra M with a faithful normal semi-finite trace τ , let $L(M, \tau)$ be the algebra of all τ -measurable operators affiliated with M . We give a complete description of all derivations on the algebra $L(M, \tau)$. In particular, we prove that if M is of type I_∞ then every derivation on $L(M, \tau)$ is inner.

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1. Introduction

The present paper is devoted to a complete description of derivations on the algebra of τ -measurable operators $L(M, \tau)$ affiliated with a type I von Neumann algebra M and a normal faithful semi-finite trace τ .

Given a (complex) algebra A , a linear operator $D : A \rightarrow A$ is called a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in A$. Each element $a \in A$ generates a derivation $D_a : A \rightarrow A$ defined as $D_a(x) = ax - xa$, $x \in A$. Such derivations are called *inner* derivations.

It is well known that all derivation on a von Neumann algebra are inner and therefore are norm continuous. But the properties of derivations on the unbounded operator algebra $L(M, \tau)$ seem to be very far from being similar. Indeed, the results of [2] and [4] show that in the commutative case where $M = L^\infty(\Omega, \Sigma, \mu)$, with (Ω, Σ, μ) any non atomic measure space with a finite measure μ , the algebra $L(M, \tau) \cong L^0(\Omega, \Sigma, \mu)$ of all complex measurable functions on (Ω, Σ, μ) admits non zero derivations. It is clear that these derivations are discontinuous in the measure topology (i. e. the topology of convergence in measure), and thus are non inner. It seems that the existence of such pathological examples deeply depends on the commutativity of the underlying algebra M . Indeed, the main result of our previous paper [1] states that if M is a type I von Neumann algebra, then any derivation D on $L(M, \tau)$, which is identically zero on the center Z of the von Neumann algebra M (i.e. D is Z -linear), is inner, i.e. $D(x) = ax - xa$ for an appropriate element $a \in L(M, \tau)$.

In the mentioned paper [1] we have also constructed an example of a non inner derivation on the algebra $L(M, \tau)$, where M is a homogeneous type I_n algebra $L^\infty(\Omega, \mu) \bar{\otimes} M_n(\mathbb{C})$. In this case $L(M, \tau)$ coincides with the algebra $M_n(L^0)$ of all $n \times n$ matrices over the algebra $L^0 = L^0(\Omega, \Sigma, \mu)$. Namely, given any non zero derivation $\delta : L^0 \rightarrow L^0$

and a matrix $(\lambda_{ij})_{i,j=1}^n \in M_n(L^0)$, $\lambda_{ij} \in L^0$, $i, j = \overline{1, n}$, put

$$D_\delta((\lambda_{ij})_{i,j=1}^n) = (\delta(\lambda_{ij})_{i,j=1}^n). \quad (1)$$

Then it is clear that D_δ defines a derivation on $M_n(L^0)$, which coincides with δ on the center of $M_n(L^0)$.

In the present paper we prove that for type I von Neumann algebras the above construction (1) gives the general form of the pathological derivations and these only exist in type I_{fin} cases, while for type I_∞ von Neumann algebras M all derivation on $L(M, \tau)$ are inner. Moreover we prove that an arbitrary derivation D on $L(M, \tau)$ for a type I von Neumann algebra M , can be uniquely decomposed into the sum $D_a + D_\delta$ where the derivation D_a is inner and the derivation D_δ is of the form (1). In such a decomposition δ is defined uniquely, and the element a is unique up to a central summand.

2. Preliminaries

Let (Ω, Σ, μ) be a measurable space and suppose that the measure μ has the direct sum property, i. e. there is a family $\{\Omega_i\}_{i \in J} \subset \Sigma$, $0 < \mu(\Omega_i) < \infty$, $i \in J$, such that for any $A \in \Sigma$, $\mu(A) < \infty$, there exist a countable subset $J_0 \subset J$ and a set B with zero measure such that $A = \bigcup_{i \in J_0} (A \cap \Omega_i) \cup B$.

We denote by $L^0 = L^0(\Omega, \Sigma, \mu)$ the algebra of all (equivalence classes of) complex measurable functions on (Ω, Σ, μ) equipped with the topology of convergence in measure. Then L^0 is a complete metrizable commutative regular algebra with the unit $\mathbf{1}$ given by $\mathbf{1}(\omega) = 1$, $\omega \in \Omega$.

Recall that a net $\{\lambda_\alpha\}$ in L^0 (o)-converges to $\lambda \in L^0$ if there exists a net $\{\xi_\alpha\}$ monotone decreasing to zero such that $|\lambda_\alpha - \lambda| \leq \xi_\alpha$ for all α .

Denote by ∇ the complete Boolean algebra of all idempotents from L^0 , i. e. $\nabla = \{\tilde{\chi}_A : A \in \Sigma\}$, where $\tilde{\chi}_A$ is the element from L^0 which contains the characteristic function of the set A . A *partition of the unit* in ∇ is a family (π_α) of orthogonal idempotents from ∇ such that $\sum_\alpha \pi_\alpha = \mathbf{1}$.

A complex linear space E is said to be normed by L^0 if there is a map $\|\cdot\| : E \longrightarrow L^0$ such that for any $x, y \in E, \lambda \in \mathbb{C}$, the following conditions are fulfilled:

- 1) $\|x\| \geq 0; \|x\| = 0 \iff x = 0$;
- 2) $\|\lambda x\| = |\lambda| \|x\|$;
- 3) $\|x + y\| \leq \|x\| + \|y\|$.

The pair $(E, \|\cdot\|)$ is called a lattice-normed space over L^0 . A lattice-normed space E is called d -decomposable, if for any $x \in E$ with $\|x\| = \lambda_1 + \lambda_2, \lambda_1, \lambda_2 \in L^0, \lambda_1 \lambda_2 = 0$, there exist $x_1, x_2 \in E$ such that $x = x_1 + x_2$ and $\|x_i\| = \lambda_i, i = 1, 2$. A net (x_α) in E is said to be *(bo)*-convergent to $x \in E$, if the net $\{\|x_\alpha - x\|\}$ *(o)*-converges to zero in L^0 . A lattice-normed space E which is d -decomposable and complete with respect to the *(bo)*-convergence is called a *Banach-Kantorovich space*.

It is known that every Banach-Kantorovich space E over L^0 is a module over L^0 and $\|\lambda x\| = |\lambda| \|x\|$ for all $\lambda \in L^0, x \in E$ (see [5]).

Let E be a Banach-Kantorovich space over L^0 . If $(u_\alpha)_{\alpha \in A}$ is a net in E and $(\pi_\alpha)_{\alpha \in A}$ is a partition of the unit in ∇ , then the series $\sum_{\alpha} \pi_\alpha u_\alpha$ *(bo)*-converges in E and its sum is called the *mixing* of $(u_\alpha)_{\alpha \in A}$ with respect to $(\pi_\alpha)_{\alpha \in A}$. We denote this sum by $\text{mix}(\pi_\alpha u_\alpha)$. A subset $K \subset E$ is called *cyclic*, if $\text{mix}(\pi_\alpha u_\alpha) \in K$ for each net $(u_\alpha)_{\alpha \in A} \subset K$ and for any partition of the unit $(\pi_\alpha)_{\alpha \in A}$ in ∇ .

Let K be a cyclic set. We say that a map $T : K \rightarrow K$ commutes with the mixing operation, if

$$T\left(\sum_{\alpha} \pi_{\alpha} x_{\alpha}\right) = \pi_{\alpha} T\left(\sum_{\alpha} x_{\alpha}\right)$$

for all $(x_\alpha) \subset L^0$ and any partition $\{\pi_\alpha\}$ of the unit in ∇ .

Let \mathcal{K} be a module over L^0 . A map $\langle \cdot, \cdot \rangle : \mathcal{K} \times \mathcal{K} \rightarrow L^0$ is called an L^0 -valued inner product, if for all $x, y, z \in \mathcal{K}, \lambda \in L^0$, the following conditions are fulfilled:

- 1) $\langle x, x \rangle \geq 0; \langle x, x \rangle = 0 \iff x = 0$;

- 2) $\langle x, y \rangle = \overline{\langle y, x \rangle}$;
- 3) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$;
- 4) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$.

If $\langle \cdot, \cdot \rangle : \mathcal{K} \times \mathcal{K} \rightarrow L^0$ is an L^0 -valued inner product, then $\|x\| = \sqrt{\langle x, x \rangle}$ defines an L^0 -valued norm on \mathcal{K} . The pair $(\mathcal{K}, \langle \cdot, \cdot \rangle)$ is called a *Kaplansky-Hilbert module* over L^0 , if $(\mathcal{K}, \|\cdot\|)$ is a Banach-Kantorovich space over L^0 (see [5]).

Let X be a Banach space. A map $s : \Omega \rightarrow X$ is said to be simple, if $s(\omega) = \sum_{k=1}^n \chi_{A_k}(\omega) c_k$, where $A_k \in \Sigma$, $A_i \cap A_j = \emptyset$, $i \neq j$, $c_k \in X$, $k = \overline{1, n}$, $n \in \mathbb{N}$. A map $u : \Omega \rightarrow X$ is said to be measurable, if there is a sequence (s_n) of simple maps such that $\|s_n(\omega) - u(\omega)\| \rightarrow 0$ almost everywhere on any $A \in \Sigma$ with $\mu(A) < \infty$.

Let $\mathcal{L}(\Omega, X)$ be the set of all measurable maps from Ω into X , and let $L^0(\Omega, X)$ denote the space of all equivalence classes with respect to the equality almost everywhere. Denote by \hat{u} the equivalence class from $L^0(\Omega, X)$ which contains the measurable map $u \in \mathcal{L}(\Omega, X)$. Further we shall identify the element $u \in \mathcal{L}(\Omega, X)$ and the class \hat{u} . Note that the function $\omega \rightarrow \|u(\omega)\|$ is measurable for any $u \in \mathcal{L}(\Omega, X)$. The equivalence class containing the function $\|u(\omega)\|$ is denoted by $\|\hat{u}\|$. For $\hat{u}, \hat{v} \in L^0(\Omega, X)$, $\lambda \in L^0$ put $\hat{u} + \hat{v} = \widehat{u(\omega) + v(\omega)}$, $\lambda \hat{u} = \widehat{\lambda(\omega)u(\omega)}$.

It is known [5] that $(L^0(\Omega, X), \|\cdot\|)$ is a Banach-Kantorovich space over L^0 .

Put $L^\infty(\Omega, X) = \{x \in L^0(\Omega, X) : \|x\| \in L^\infty(\Omega)\}$. Then $L^\infty(\Omega, X)$ is a Banach space with respect to the norm $\|x\|_\infty = \|\|x\|\|_{L^\infty(\Omega)}$.

If H is a Hilbert space, then $L^0(\Omega, H)$ can be equipped with an L^0 -valued inner product $\langle x, y \rangle = \widehat{(x(\omega), y(\omega))}$, where (\cdot, \cdot) is the inner product on H .

Then $(L^0(\Omega, H), \langle \cdot, \cdot \rangle)$ is a Kaplansky-Hilbert module over L^0 .

Let E be a Banach-Kantorovich space over L^0 . An operator $T : E \rightarrow E$ is L^0 -linear if $T(\lambda_1 x_1 + \lambda_2 x_2) = \lambda_1 T(x_1) + \lambda_2 T(x_2)$ for all $\lambda_1, \lambda_2 \in L^0$.

L^0 , $x_1, x_2 \in E$. An L^0 -linear operator $T : E \rightarrow E$ is L^0 -bounded if there exists an element $c \in L^0$ such that $\|T(x)\| \leq c\|x\|$ for any $x \in E$. For an L^0 -bounded L^0 -linear operator T we put $\|T\| = \sup\{\|T(x)\| : \|x\| \leq \mathbf{1}\}$.

Let $B(H)$ be the algebra of all bounded linear operators on a Hilbert space H and let M be a von Neumann algebra in $B(H)$ with a faithful normal semi-finite trace τ . Denote by $\mathcal{P}(M)$ the lattice of projections in M .

A linear subspace \mathcal{D} in H is said to be affiliated with M (denoted as $\mathcal{D}\eta M$), if $u(\mathcal{D}) \subset \mathcal{D}$ for any unitary operator u from the commutant

$$M' = \{y \in B(H) : xy = yx, \forall x \in M\}$$

of the algebra M .

A linear operator x on H with the domain $\mathcal{D}(x)$ is said to be affiliated with M (denoted as $x\eta M$) if $u(\mathcal{D}(x)) \subset \mathcal{D}(x)$ and $ux(\xi) = xu(\xi)$ for all $u \in M'$, $\xi \in \mathcal{D}(x)$.

A linear subspace \mathcal{D} in H is called τ -dense, if

- 1) $\mathcal{D}\eta M$;
- 2) given any $\varepsilon > 0$ there exists a projection $p \in \mathcal{P}(M)$ such that $p(H) \subset \mathcal{D}$ and $\tau(p^\perp) \leq \varepsilon$.

A closed linear operator x is said to be τ -measurable (or totally measurable) with respect to the von Neumann algebra M , if $x\eta M$ and $\mathcal{D}(x)$ is τ -dense in H .

Denote by $L(M, \tau)$ the set of all τ -measurable operators with respect to M . Let $\|\cdot\|_M$ stand for the uniform norm in M . The *measure topology*, t_τ in $L(M, \tau)$ is the one given by the following system of neighborhoods of zero:

$$V(\varepsilon, \delta) = \{x \in L(M, \tau) : \exists e \in \mathcal{P}(M), \tau(e^\perp) \leq \delta, xe \in M, \|xe\|_M \leq \varepsilon\},$$

where $\varepsilon > 0, \delta > 0$.

It is known [7] that $L(M, \tau)$ equipped with the measure topology is a complete metrizable topological $*$ -algebra.

Let $L^\infty(\Omega, \mu) \bar{\otimes} B(H)$ be the tensor product of von Neumann algebras $L^\infty(\Omega, \mu)$ and $B(H)$, with the trace $\tau = \mu \otimes Tr$, where Tr is the canonical trace for operators in $B(H)$ (with its natural domain).

Denote by $L_t^0(\Omega, B(H))$ the space of equivalence classes of point-wise Bochner measurable operator-valued maps from Ω into $B(H)$ (see [8]).

Given $\hat{u}, \hat{v} \in L_t^0(\Omega, B(H))$ put $\hat{u}\hat{v} = \widehat{u(\omega)v(\omega)}$, $\hat{u}^* = \widehat{u(\omega)^*}$.

Define

$$L_t^\infty(\Omega, B(H)) = \{x \in L_t^0(\Omega, B(H)) : \|x\| \in L^\infty(\Omega)\}.$$

The space $(L_t^\infty(\Omega, B(H)), \|\cdot\|_\infty)$ is a Banach $*$ -algebra.

It is known [8] that the algebra $L^\infty(\Omega, \mu) \bar{\otimes} B(H)$ is $*$ -isomorphic with the algebra $L_t^\infty(\Omega, B(H))$.

Note also that

$$\tau(x) = \int_{\Omega} Tr(x(\omega)) d\mu(\omega).$$

Further we shall identify the algebra $L^\infty(\Omega, \mu) \bar{\otimes} B(H)$ with the algebra $L_t^\infty(\Omega, B(H))$.

Denote by $B(L^0(\Omega, H))$ the algebra of all L^0 -linear and L^0 -bounded operators on $L^0(\Omega, H)$.

Given any $f \in L_t^0(\Omega, B(H))$ consider the element $\Psi(f)$ from $B(L^0(\Omega, H))$ defined by

$$\Psi(f)(x) = \widehat{f(\omega)(x(\omega))}, \quad x \in L^0(\Omega, H).$$

Then the correspondence $f \rightarrow \Psi(f)$ gives an isometric $*$ -isomorphism between the algebras $L_t^0(\Omega, B(H))$ and $B(L^0(\Omega, H))$ (see [5]).

It is known [1], that the algebra $L(L^\infty(\Omega, \mu) \bar{\otimes} B(H), \tau)$ of all τ -measurable operators with respect to $L^\infty(\Omega, \mu) \bar{\otimes} B(H)$ is $*$ -isomorphic with the algebra $L_t^0(\Omega, B(H))$.

Therefore one has the following relations for the algebras mentioned above:

$$L(L^\infty(\Omega) \bar{\otimes} B(H), \tau) \cong L_t^0(\Omega, B(H)) \cong B(L^0(\Omega, H)) \quad (2)$$

(\cong standing for $*$ -isomorphic).

The above isomorphisms enable us to obtain the following necessary and sufficient condition for a derivation on the algebra $L(M, \tau)$ to be inner.

Theorem 2.1 [1]. *Let M be a type I von Neumann algebra with the center Z . A derivation D on the algebra $L(M, \tau)$ is inner if and only if it is Z -linear, or equivalently, it is identically zero on Z .*

3. Main results

Let A be an algebra with the center Z and let $D : A \rightarrow A$ be a derivation. Given any $x \in A$ and a central element $z \in Z$ we have

$$D(zx) = D(z)x + zD(x)$$

and

$$D(xz) = D(x)z + xD(z).$$

Since $zx = xz$ and $zD(x) = D(x)z$, it follows that $D(z)x = xD(z)$ for any $x \in A$. This means that $D(z) \in Z$, i.e. $D(Z) \subseteq Z$. Therefore given any derivation D on the algebra A we can consider its restriction δ onto the center Z :

$$\delta : z \rightarrow D(z), z \in Z.$$

This simple but important remark is crucial in our further considerations.

Let M be a homogeneous von Neumann algebra of type I. Then [8] M is isomorphic to the tensor product $L^\infty(\Omega) \bar{\otimes} B(H)$.

First, let us consider the case $\dim H = n < \infty$. In this case $B(H)$ coincides with the algebra of $M_n(\mathbb{C})$ of $n \times n$ complex matrices, and from (2) we have that $L(M, \tau) \cong B(L^0(\Omega, H))$ is isomorphic with the algebra $M_n(L^0)$ of all $n \times n$ matrices over the algebra L^0 .

In the papers [2], [4] the existence of non zero derivations on L^0 has been proved in the case of a non atomic measure space (Ω, Σ, μ) . Given

any derivation $\delta : L^0 \rightarrow L^0$ consider the elementwise derivation D_δ on $M_n(L^0)$ defined as (1):

$$D_\delta((\lambda_{ij})_{i,j=1}^n) = (\delta(\lambda_{ij})_{i,j=1}^n),$$

where $(\lambda_{ij})_{i,j=1}^n \in M_n(L^0)$.

A straightforward calculation shows that D_δ is indeed a derivation on $M_n(L^0)$ and its restriction onto the center of $M_n(L^0)$ coincides with δ (recall that the center of $M_n(L^0)$ is isomorphic with L^0).

Lemma 3.1. *Any derivation D on the algebra $M_n(L^0)$ admits a unique decomposition*

$$D = D_a + D_\delta,$$

where D_a is an inner derivation and D_δ is a derivation of the form (1).

Proof. Given a derivation D on $M_n(L^0)$, consider its restriction δ onto the center $Z = L^0$ and extend it to the whole $M_n(L^0)$ by the form (1) as D_δ . Put $D_1 = D - D_\delta$. Then given any $z \in Z$ we have

$$D_1(z) = D(z) - D_\delta(z) = D(z) - D(z) = 0,$$

i.e. D_1 is identically zero on Z and therefore it is Z -linear. Theorem 2.1 implies that D_1 is inner, i.e. $D_1 = D_a$ for an appropriate $a \in M_n(L^0)$. Therefore $D = D_a + D_\delta$.

Now suppose that

$$D = D_{a_1} + D_{\delta_1} = D_{a_2} + D_{\delta_2}.$$

Then $D_{a_1} - D_{a_2} = D_{\delta_2} - D_{\delta_1}$. Since $D_{a_1} - D_{a_2}$ is identically zero on the center of $M_n(L^0)$, then $D_{\delta_2} - D_{\delta_1}$ also is identically zero on the center of $M_n(L^0)$. Thus $\delta_1 = \delta_2$ and hence $D_{a_1} = D_{a_2}$. The proof is complete. ■

In order to consider the case of homogeneous type I von Neumann algebra $L^\infty(\Omega) \bar{\otimes} B(H)$ with $\dim H = \infty$, we need some auxiliary results.

Lemma 3.2. *Any derivation δ on the algebra L^0 commutes with the mixing operation on nets in L^0 .*

Proof. Consider a net $\{\lambda_\alpha\}$ in L^0 and a partition of the unit $\{\pi_\alpha\}$ in $\nabla \subset L^0$. Since $\delta(\pi) = 0$ for any idempotent $\pi \in \nabla$, we have $\delta(\pi_\alpha) = 0$ for all α and thus $\delta(\pi_\alpha \lambda) = \pi_\alpha \delta(\lambda)$ for any $\lambda \in L^0$. Therefore for each π_{α_0} from the given partition of the unit we have

$$\pi_{\alpha_0} \delta\left(\sum_{\alpha} \pi_{\alpha} \lambda_{\alpha}\right) = \delta\left(\pi_{\alpha_0} \sum_{\alpha} \pi_{\alpha} \lambda_{\alpha}\right) = \delta(\pi_{\alpha_0} \lambda_{\alpha_0}) = \pi_{\alpha_0} \delta(\lambda_{\alpha_0}).$$

By taking the sum over all α_0 we obtain

$$\delta\left(\sum_{\alpha} \pi_{\alpha} \lambda_{\alpha}\right) = \sum_{\alpha} \pi_{\alpha} \delta(\lambda_{\alpha}).$$

The proof is complete. ■

Lemma 3.3. *Given any non trivial derivation $\delta : L^0 \rightarrow L^0$ there exist a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in L^0 with $|\lambda_n| \leq \mathbf{1}$, $n \in \mathbb{N}$, and an idempotent $\pi \in \nabla$, $\pi \neq 0$ such that*

$$|\delta(\lambda_n)| \geq n\pi$$

for all $n \in \mathbb{N}$.

Proof. Suppose that the set $\{\delta(\lambda) : \lambda \in L^0, |\lambda| \leq \mathbf{1}\}$ is order bounded in L^0 . Then δ maps any uniformly convergent sequence in $L^{\infty}(\Omega)$ to an (o) -convergent sequence in L^0 . The algebra $L^{\infty}(\Omega)$ coincides with the uniform closure of the linear span of idempotents from ∇ . Since δ is identically zero on ∇ it follows that $\delta \equiv 0$ on $L^{\infty}(\Omega)$. Since δ commutes with the mixing operation and any element $\lambda \in L^0$ can be represented as $\lambda = \sum_{\alpha} \pi_{\alpha} \lambda_{\alpha}$, where $\{\lambda_{\alpha}\} \subset L^{\infty}(\Omega, \mu)$, and $\{\pi_{\alpha}\}$ is a partition of unit in ∇ , we have $\delta(\lambda) = \delta\left(\sum_{\alpha} \pi_{\alpha} \lambda_{\alpha}\right) = \sum_{\alpha} \pi_{\alpha} \delta(\lambda_{\alpha}) = 0$, i.e. $\delta \equiv 0$ on L^0 . This contradiction shows that the set $\{\delta(\lambda) : \lambda \in L^0, |\lambda| \leq \mathbf{1}\}$ is not order bounded in L^0 . Further, since δ commutes with the mixing operations and the set $\{\lambda : \lambda \in L^0, |\lambda| \leq \mathbf{1}\}$ is cyclic, the set $\{\delta(\lambda) : \lambda \in L^0, |\lambda| \leq \mathbf{1}\}$ is also cyclic. By [3, Proposition 3] there exist a sequence $\{\lambda_n\}_{n=1}^{\infty}$ in L^0 and an idempotent $\pi \in \nabla$, $\pi \neq 0$, such that $|\delta(\lambda_n)| \geq n\pi$, $n \in \mathbb{N}$. The proof is complete. ■

Now we are in position to consider derivations on measurable operators for homogeneous von Neumann algebras of type I_∞ .

Theorem 3.4. *If $\dim H = \infty$, then any derivation on the algebra $L(L^\infty(\Omega) \bar{\otimes} B(H), \tau)$ is inner.*

Proof. First let us suppose that H is separable and let $\{\varphi_n\}_{n=1}^\infty$ be an orthonormal basis in H . Given any $n \in \mathbb{N}$ put $e_n = \mathbf{1} \otimes \varphi_n$, i.e. e_n is the equivalence class in $L^0(\Omega, H)$ which contains the constant mapping $\omega \rightarrow \varphi_n$.

Then $\{e_n\}_{n=1}^\infty$ is a ∇ -orthonormal basis in $L^0(\Omega, H)$, i.e. $\langle e_i, e_j \rangle = \delta_{ij} \mathbf{1}$, where δ_{ij} is the Kronecker symbol, and any element $\xi \in L^0(\Omega, H)$ has the form

$$\xi = \sum_{k=1}^{\infty} \alpha_k e_k,$$

where $\alpha_k \in L^0$, $\sum_{k=1}^{\infty} |\alpha_k|^2 \in L^0$.

For each $n \in \mathbb{N}$ consider the orthogonal projection p_n onto the submodule $\{\alpha e_n : \alpha \in L^0\} \subset L^0(\Omega, H)$, i.e.

$$p_n\left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \alpha_n e_n.$$

For any order bounded sequence $\Lambda = \{\lambda_k\}$ in L^0 define an operator x_Λ on $B(L^0(\Omega, H))$ putting

$$x_\Lambda\left(\sum_{k=1}^{\infty} \alpha_k e_k\right) = \sum_{k=1}^{\infty} \lambda_k \alpha_k e_k.$$

Then

$$x_\Lambda p_n = p_n x_\Lambda = \lambda_n p_n. \tag{3}$$

Let D be a derivation on $B(L^0(\Omega, H))$, and let δ be its restriction onto the center of $B(L^0(\Omega, H))$, identified with L^0 .

Take any $\lambda \in L^0$ and $n \in \mathbb{N}$. From the identity

$$D(\lambda p_n) = D(\lambda) p_n + \lambda D(p_n)$$

multiplying by p_n on both sides we obtain

$$p_n D(\lambda p_n) p_n = p_n D(\lambda) p_n + \lambda p_n D(p_n) p_n.$$

Since p_n is a projection, one has that $p_n D(p_n) p_n = 0$, and since $D(\lambda) = \delta(\lambda) \in Z$, we have

$$p_n D(\lambda p_n) p_n = \delta(\lambda) p_n. \quad (4)$$

Now from the identity

$$D(x_\Lambda p_n) = D(x_\Lambda) p_n + x_\Lambda D(p_n),$$

in view of (3) one has similarly

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n + \lambda p_n D(p_n) p_n,$$

i.e.

$$p_n D(\lambda_n p_n) p_n = p_n D(x_\Lambda) p_n. \quad (5)$$

Now (4) and (5) imply

$$p_n D(x_\Lambda) p_n = \delta(\lambda_n) p_n.$$

Further we have

$$\|p_n D(x_\Lambda) p_n\| \leq \|p_n\| \|D(x_\Lambda)\| \|p_n\| = \|D(x_\Lambda)\|$$

and

$$\|\delta(\lambda_n) p_n\| = |\delta(\lambda_n)|.$$

Therefore

$$\|D(x_\Lambda)\| \geq |\delta(\lambda_n)|$$

for any bounded sequence $\Lambda = \{\lambda_n\}$ in L^0 .

If we suppose that $\delta \neq 0$ then by Lemma 3.3 there exist a bounded sequence $\Lambda = \{\lambda_n\}$ in L^0 and an idempotent $\pi \in \nabla$, $\pi \neq 0$, such that

$$|\delta(\lambda_n)| \geq n\pi$$

for any $n \in \mathbb{N}$. Thus

$$\|D(x_\Lambda)\| \geq n\pi$$

for all $n \in \mathbb{N}$, i.e. $\pi = 0$ – which is a contradiction. Therefore $\delta \equiv 0$, i.e. D is identically zero on the center of $B(L^0(\Omega, H))$. By Theorem 2.1 D is inner.

Now suppose that H is not necessarily separable and take a projection p in $B(H)$ such that $p(H)$ is a separable infinite dimensional Hilbert space. Put $e = \mathbf{1} \otimes p$. Define a derivation D_e on $eB(L^0(\Omega, H))e$ by putting

$$D_e(x) = eD(x)e, \quad x \in eB(L^0(\Omega, H))e.$$

Since the algebra $eB(L^0(\Omega, H))e$ is isomorphic with $B(L^0(\Omega, p(H)))$ we have from the above separable case that D_e is zero on the center of $eB(L^0(\Omega, H))e = L^0e$. Therefore one has $0 = eD(\lambda e)e = eD(\lambda)e + \lambda eD(e)e = eD(\lambda)e = D(\lambda)e$, i.e. $D(\lambda) = 0$ for all $\lambda \in L^0$. Thus D is identically zero on the center of $B(L^0(\Omega, H))$. By Theorem 2.1 D is inner. The proof is complete. ■

Now let us consider the general case of type I von Neumann algebras.

Recall that a von Neumann algebra M is of *type I* if it is isomorphic to a von Neumann algebra with an abelian commutant, or, equivalently M admits a faithful abelian projection.

It is well-known [8] that if M is a type I von Neumann algebra then there is a unique (cardinal-indexed) orthogonal family of projections $(q_\alpha)_{\alpha \in I} \subset \mathcal{P}(M)$ with $\sum_{\alpha \in I} q_\alpha = \mathbf{1}$ such that $q_\alpha M$ is a homogeneous type I_α von Neumann algebra and it is isomorphic to the tensor product of an abelian von Neumann algebra $L^\infty(\Omega_\alpha, \mu_\alpha)$ and $B(H_\alpha)$ with $\dim H_\alpha = \alpha$, i. e.

$$M \cong \sum_{\alpha}^{\oplus} L^\infty(\Omega_\alpha, \mu_\alpha) \bar{\otimes} B(H_\alpha).$$

Consider the faithful normal semi-finite trace τ on M , defined by

$$\tau(x) = \sum_{\alpha} \tau_\alpha(x_\alpha), \quad x = (x_\alpha) \in M, \quad x \geq 0,$$

where $\tau_\alpha = \mu_\alpha \otimes Tr_\alpha$.

Let

$$\prod_{\alpha} L(L^{\infty}(\Omega_{\alpha}, \mu_{\alpha}) \bar{\otimes} B(H_{\alpha}), \tau_{\alpha})$$

be the topological (Tychonoff) product of the spaces $L(L^{\infty}(\Omega_{\alpha}, \mu_{\alpha}) \bar{\otimes} B(H_{\alpha}), \tau_{\alpha})$ equipped with corresponding measure topologies.

Then (see [6]) we have the topological embedding

$$L(M, \tau) \subset \prod_{\alpha} L(L^{\infty}(\Omega_{\alpha}, \mu_{\alpha}) \bar{\otimes} B(H_{\alpha}), \tau_{\alpha}).$$

It should be noted that the above topological imbedding is strict in general (see [6]).

Now let δ be a derivation on the center Z of $L(M, \tau)$. Since $q_{\alpha}Z \cong L^0(\Omega_{\alpha})$ for each α , and δ maps each $q_{\alpha}Z$ into $q_{\alpha}Z$, it follows that δ induces a derivation δ_{α} on each $L^0(\Omega_{\alpha})$.

Put $F = \{\alpha \in I : \dim H_{\alpha} = \alpha < \infty\}$. Let $D_{\delta_{\alpha}}$ ($\alpha \in F$) be the derivation on the matrix algebra $q_{\alpha}L(M, \tau) \cong M_{\alpha}(L^0(\Omega_{\alpha}))$ constructed by formula (1). For $\alpha \in I \setminus F$ put $D_{\delta_{\alpha}} \equiv 0$. Now define a derivation D_{δ} on $L(M, \tau)$ by putting

$$D_{\delta}(x) = (D_{\delta_{\alpha}}(x_{\alpha})), \quad x = (x_{\alpha}) \in L(M, \tau). \quad (6)$$

From Lemma 3.1 and Theorem 3.4 we obtain the following main result of the paper.

Theorem 3.5. *Let M be a type I von Neumann algebra. Any derivation D on the algebra $L(M, \tau)$ can be decomposed in a unique way as*

$$D = D_a + D_{\delta},$$

where D_a is an inner derivation and D_{δ} is a derivation of the form (6).

Corollary 3.6. *Let M be a von Neumann algebra of type I_{∞} . Then any derivation on the algebra $L(M, \tau)$ is inner.*

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References

1. S. Albeverio, Sh. A. Ayupov, K. K. Kudaybergenov, Derivations on the Algebra of Measurable Operators Affiliated with a Type I von Neumann Algebra // SFB 611, Universität Bonn, Preprint, N 301, 2006. arXivmath.OA/0703171v1.
2. A. F. Ber, V. I. Chilin, F. A. Sukochev, Non-trivial derivation on commutative regular algebras. *Extracta mathematicae*, 21 (2006), No 2, 107-147.
3. I. G. Ganiev, K. K. Kudaybergenov, The Banach-Steinhaus uniform boundedness principle for operators in Banach-Kantorovich spaces over L^0 . *Siberian Adv. Math.* 16 (2006), No. 3, 42–53.
4. A. G. Kusraev, Automorphisms and Derivations on a Universally Complete Complex f-Algebra, *Sib. Math. Jour.* 47 (2006), No 1, 77-85.
5. A. G. Kusraev, *Dominated Operators*, Kluwer Academic Publishers, Dordrecht, 2000.
6. M. A. Muratov, V. I. Chilin, $*$ -algebras of unbounded operators affiliated with a von Neumann algebra, *J. Math. Sciences*, 140 (2007), No. 3, 445-451.
7. E. Nelson, Notes on non-commutative integration, *J. Funct. Anal.*, 15 (1975), 91-102.
8. M. Takesaki, *Theory of Operator Algebras. I*, Springer-Verlag, New York; Heidelberg; Berlin (1991).